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SHOCK PROPAGATION AND ATTENUATION IN BUBBLY LIQUIDS: MODELING WAVE PROPAGATION USING A NONLINEAR EQUATION-OF-STATE

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Executive Summary — Bubbly media play a significant role in underwater acoustics, medical ultrasound and in industrial systems where gas-liquid flows are present. The focus of our research has been to develop a continuum model for bubbly mixtures that can be used to model physical phenomena in these areas. The key to the continuum model is a nonlinear, non-equilibrium equation of state (EOS) that relates pressure to the mixture density and the number density (number of bubbles per unit volume) and their first two material time derivatives. The derivation of the EOS is presented here and a number of traveling wave solutions obtained using this nonlinear EOS are discussed. To develop an accurate model, two important damping mechanisms for the medium had to be incorporated: heat transfer and relative motion between the gas and liquid phases. To quantify the importance of heat transfer, an analysis of single-bubble radial oscillations was completed in this work, and a Padé approximation for the thermal damping was derived from the linearized gas dynamics equations. A second important damping mechanism arises from relative motion between the gas bubbles and the liquid. The quantitative effects of relative motion on the damping of waves in bubbly liquids has also been examined and is described here.

Contents

1 Introduction

The study of wave propagation in liquids containing gas bubbles is an interesting multiphase flow problem because the bubbles themselves can behave in a highly nonlinear fashion due to their large compressibility. Bubbly liquids occur in a number of environmental and industrial settings. Bubbles are present near the surface of the ocean in the form of bubble clouds and plumes. Air becomes entrained in the ocean by breaking waves or the passing of surface ships. Bubbles oscillations give rise to underwater sound or, in the case of surface ships, collapse of cavitation bubbles may cause damage to propeller blades. In industry, the presence of bubbly mixtures can lead to inefficient performance in chemical reactors and boilers, or to safety problems for nuclear power plants. Bubbles also play a role in applications of medical ultrasound, such as lithotrispy (breaking of kidney stones by sound) and through their use as contrast agents during ultrasonic imaging. The objectives of this work were to devise a continuum-level description of wave propagation and fluid flow in bubbly liquids and to study nonlinear waves (e.g. shocks) in such media.

Campbell and Pitcher [1] carried out the first qualitative experiments on shock waves in bubbly liquids. They established that the Mach number of the wave must be greater than unity for shocks to exist. The Mach number was defined as the speed at which the shock propagates divided by the sound speed of the medium ahead of the shock. Experiments performed by Noordzij and van Wijngaarden [2] depicted three qualitatively distinct shock waveforms by stages. The first stage is considered the "classical" shock profile. This profile can be characterized by a steep rise in pressure from a low equilibrium value followed by an oscillatory relaxation region where the waveform oscillates about a high equilibrium value. The oscillations that follow the initial discontinuity are a direct result of the presence of gas bubbles in the liquid.

Van Wijngaarden [3] was the first investigator to present a theoretical model describing bubbly liquids as a continuum. The derivation of the model was primarily based on physical arguments and so the extent to which the equations were valid was not rigorously defined. Later, Caflisch et al. [4] started with the equations describing the medium on a microscopic scale, and through the use of asymptotic homogenization, derived equations valid on the macroscopic scale. Further theoretical work was done to incorporate additional physical effects into the model. Watanabe and Prosperetti [5] included thermal conduction, which is considered the most dominant damping mechanism during bubble oscillations. Most often the bubbles are assumed to expand and contract adiabatically, but this is known to be inaccurate in practice. Sangani [6] extended the model to include bubble-bubble interactions. Tan and Bankoff [7] considered theoretically the effects of relative motion between the two phases. Their work was followed up a decade later by Kameda and Matsumoto [8] and by Ishii et al. [9] In these papers, the authors were able to incorporate the effects of relative motion as well as some thermal effects into their numerical simulations. However, their results were obtained by performing calculations on each bubble in the flow, negating the advantage of considering a bubbly medium as a continuum. More recently, Kameda and Matsumoto [10] have shown that there can be good agreement between theory and experiments, if the experiments are performed in bubbly liquids in which the bubble distribution is spatially uniform. There is an extensive list of work focussing on

shock propagation in bubbly liquids which includes [11]–[27].

In order for a continuum description of bubbly liquids to be valid, there needs to exist a separation of length scales. Typically one assumes that the distance between bubbles is much larger than the radii of the bubbles, so bubble-bubble interactions can be neglected. The size of an averaging volume should be large so that it contains many bubbles, but small when compared to the acoustic wavelengths of interest. With the latter assumption, average field quantities (e.g. pressure) can be defined in the averaging volume, which are spatially uniform over the volume. As a consequence of this separation of length scales, a non-equilibrium equation of state (EOS) can be derived. For flows where the relative motion between the phases is negligible, this EOS is a nonlinear relation between the pressure and the density and its first two material time derivatives. Nigmatulin [28] was the first to point out that an EOS of this form can be obtained; wave motion in such media was further investigated by Gavrilyuk [29]. Other assumptions for the derivation of the EOS are liquid incompressibility, monodispersity, and absence of bubble breakup or coalescence.

With the aim of describing shock propagation in bubbly liquids quantitatively, we present several ideas for theoretical modeling and discuss preliminary results. We first consider the issue of thermal damping of a single bubble in an infinite liquid. Insightful work investigating thermal damping in single bubble oscillations can be found in papers by Devin [30] and Prosperetti [31]. Solutions for the equation of motion for an isothermal or an adiabatic bubble are straightforward but not very accurate in modeling real bubbles. By accurately describing heat conduction for a linearly oscillating bubble, we develop an effective thermal viscosity which will contribute to the damping of sound waves in bubbly liquids. Values of this viscosity are found to be only a function of the equilibrium radius of the bubble and can be written in the form of a Padé approximate.

In the next phase of our analysis we derive the non-equilibrium EOS for a monodisperse bubbly liquid incorporating the relative motion between the gas phase and the liquid phase. The importance of the EOS is that it contains all the physics needed to characterize the nonlinearity of the medium in a continuum model. This approach is meritorious both theoretically and computationally for its simplicity. We will not have to keep track of each individual bubble in order to calculate nonlinear wave propagation through the bubbly medium. Including relative motion into the model leads to a greater understanding of the damping mechanisms, and how they interact with one another, within bubbly liquids. Also presented herein are some results which are obtained through this approach. By seeking a traveling wave solution using the EOS in conjunction with the fully nonlinear continuity and Euler equations, a phase-plane description of the system can be obtained. The phase portrait provides insights into the qualitative behavior of shocks in bubbly liquids.

2 Thermal Damping of a Single Bubble: Padé Approximation

The derivation for the thermal damping of a single oscillating bubble begins with the Rayleigh-Plesset equation. The Rayleigh-Plesset equation describes the nonlinear motion of a radially oscillating

bubble in an infinite liquid. It is a second order ordinary differential equation (ODE) for the radius $R(t)$ of the bubble as a function of time t:

$$\rho_\ell[R\ddot{R} + \frac{3}{2}\dot{R}^2] + 4\mu_\ell\frac{\dot{R}}{R} = P_{\text{gw}} - P^\infty(t)\,. \tag{1}$$

Here ρ_ℓ and μ_ℓ are the density and viscosity of the liquid, P^∞ is the forcing pressure far from the bubble center and, for this derivation, it is considered to be known. P_{gw} is the pressure inside the bubble evaluated at the bubble wall. In order to solve Eqn. (1), the pressure inside the bubble must be determined. For this purpose the equations of gas dynamics within the bubble need to be solved. They are:

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0 \tag{2}$$

$$\rho\frac{D\mathbf{v}}{Dt} + \nabla P = 0 \tag{3}$$

$$\rho c_p\frac{DT}{Dt} - \frac{DP}{Dt} = \nabla\cdot[\kappa\nabla T] \tag{4}$$

$$P = \rho\mathcal{R}T \tag{5}$$

representing the conservations of mass, momentum, and energy and the equation of state for an ideal gas, respectively. In the above equations ρ, P, $\mathbf{v}$ and T are the density, pressure, velocity and temperature of the gas. Whereas κ is the thermal conductivity, c_p is the specific heat capacity of the gas at constant pressure, and $\mathcal{R}$ is the universal gas constant. Note that the viscosity of the gas is neglected in this analysis whereas thermal conduction is included. We now assume a perturbation expansion of Eqns. (1)–(5) in the following form

$$\begin{aligned}
\rho &= \rho_0 + \epsilon\rho_1 + \ldots \\
P &= P_0 + \epsilon P_1 + \ldots \\
\mathbf{v} &= 0 + \epsilon\mathbf{v}_1 + \ldots \\
T &= T_0 + \epsilon T_1 + \ldots \\
R &= R_0 + \epsilon R_1 + \ldots
\end{aligned}$$

The $\mathcal{O}(\epsilon)$ equations in the radial direction are

$$\frac{\partial\rho_1}{\partial t} + \frac{\rho_0}{r^2}\frac{\partial}{\partial r}(r^2 v_1) = 0 \tag{6}$$

$$\rho_0\frac{\partial v_1}{\partial t} + \frac{\partial P_1}{\partial r} = 0 \tag{7}$$

$$\rho_0 c_p\frac{\partial T_1}{\partial t} - \frac{\partial P_1}{\partial t} = \frac{\kappa}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_1}{\partial r}\right) \tag{8}$$

$$P_1 = \mathcal{R}(T_0\rho_1 + \rho_0 T_1) \tag{9}$$

with the Rayleigh-Plesset written as

$$\rho_\ell R_0 \ddot{R}_1 + 4\mu_\ell \frac{\dot{R}_1}{R_0} - P_1|_{r=1} = -P^\infty(t) \tag{10}$$

We scale the variables as follows

$$r^* = \frac{r}{R_0} \qquad P_1^* = \frac{P_1}{P_0}$$

$$t^* = \omega_o t \qquad \rho_1^* = \frac{\rho_1}{\rho_0}$$

$$T_1^* = \frac{T_1}{T_0} \qquad v_1^* = \frac{v_1}{R_0\omega_o}$$

The time scale is based on the period of oscillation for a single adiabatic bubble whose natural frequency can be written as

$$\omega_o = \sqrt{\frac{3\gamma P_0}{\rho_\ell R_0^2}}.$$

Rewriting the $\mathcal{O}(\epsilon)$ equations in dimensionless form (dropping the *'s) yields

$$\frac{\partial \rho_1}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_1) = 0 \tag{11}$$

$$\frac{\partial v_1}{\partial t} + \frac{1}{M_b^2}\frac{\partial P_1}{\partial r} = 0 \tag{12}$$

$$\frac{\partial T_1}{\partial t} - (\frac{\gamma-1}{\gamma})\frac{\partial P_1}{\partial t} = \frac{1}{\mathrm{Pe}}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_1}{\partial r}\right) \tag{13}$$

$$P_1 = (\rho_1 + T_1) \tag{14}$$

In Eqns. (11)-(14) we have introduced two key nondimensional parameters, the Mach number and the Peclet number:

$$M_b = \omega_o R_0 \sqrt{\frac{\rho_0}{P_0}} \qquad \mathrm{Pe} = \frac{\omega_o R_0^2}{\kappa/\rho c_p}$$

The Peclet number Pe is a measure of convective to diffusive effects. If the Peclet number is small, there is a thick thermal boundary layer in the bubble and the bubble can be considered isothermal. For large Peclet numbers, on the other hand, the bubble has a thin thermal boundary layer and can be considered adiabatic. The second dimensionless parameter is the acoustic Mach number, M_b, inside the bubble. For an ideal gas inside an oscillating bubble, the acoustic Mach number is small. Therefore, the r-momentum equation reduces to

$$\frac{\partial P_1}{\partial r} = 0 \qquad \longrightarrow \qquad P_1 = P_1(t)$$

We add the continuity and energy equations and use the ideal gas equation of state Eqn. (14) to get

$$\frac{1}{\gamma}\frac{dP_1}{dt} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_1) = \frac{1}{\mathrm{Pe}}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_1}{\partial r}\right) \tag{15}$$

Multiply Eqn. (15) by r^2 and integrating from 0 to 1 yields

$$\left.\frac{r^3}{3\gamma}\frac{dP_1}{dt}\right|_0^1 + \left.r^2 v_1\right|_0^1 = \left.\frac{1}{\mathrm{Pe}}\, r^2\frac{\partial T_1}{\partial r}\right|_0^1 . \tag{16}$$

Eqn. (16) gives an expression for time derivative of the pressure in terms of the heat flux at the surface:

$$\frac{dP_1}{dt} = -3\gamma\dot{R}_1 + \left.\frac{3\gamma}{\mathrm{Pe}}\frac{\partial T_1}{\partial r}\right|_{r=1} \tag{17}$$

Equation (17) can now be used in the energy equation (13), resulting in an equation for the temperature field only:

$$\frac{\partial T_1}{\partial t} + 3(\gamma-1)\left(\dot{R}_1 - \left.\frac{1}{\mathrm{Pe}}\frac{\partial T_1}{\partial r}\right|_{r=1}\right) = \frac{1}{\mathrm{Pe}}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_1}{\partial r}\right) . \tag{18}$$

Since the gas is in thermal equilibrium with the liquid and the temperature of the liquid is a constant, T_0, we have the following boundary for T_1 for the equation above

$$T_1 = 0 \qquad \text{at} \quad r = 1 . \tag{19}$$

We now seek solutions to (18) of constant frequency

$$P_1 = \Re[\tilde{P}e^{i\Omega t}]$$

$$T_1 = \Re[\tilde{T}(r)e^{i\Omega t}]$$

$$R_1 = \Re[\tilde{R}e^{i\Omega t}]$$

Eqn. (18) becomes

$$i\Omega\tilde{T} + 3(\gamma-1)\left[i\Omega\tilde{R} - \frac{1}{\mathrm{Pe}}\tilde{T}'(1)\right] = \frac{1}{\mathrm{Pe}}\frac{1}{r^2}\frac{d}{dr}(r^2\tilde{T}') \tag{20}$$

Since the second term in Eqn. (20) is a constant, we define a new variable, Θ, which takes this into account

$$\Theta = \tilde{T} + 3(\gamma-1)\left[\tilde{R} + \frac{i}{\Omega\mathrm{Pe}}\tilde{T}'(1)\right] . \tag{21}$$

Now we have a much simpler ODE to solve, namely

$$i\Omega\,\Theta(r) = \frac{1}{\mathrm{Pe}}\frac{1}{r^2}\frac{d}{dr}(r^2\Theta') , \tag{22}$$

with the boundary conditions

$$\Theta(1) = 3(\gamma-1)\left[\tilde{R} + \frac{i}{\Omega \mathrm{Pe}}\tilde{T}'(1)\right]$$

$$\Theta(0) \qquad \text{is finite}$$

For differential equations of this form, rewriting the dependent variable, Θ, as the ratio $\frac{Q(r)}{r}$ can lead to the simplified ODE

$$2i\lambda\, Q(r) = Q''(r) \tag{23}$$

in which

$$\lambda^2 = \frac{\Omega \mathrm{Pe}}{2}$$

with new boundary conditions

$$Q(1) = 3(\gamma-1)\left[\tilde{R} + \frac{i}{2\lambda^2}\tilde{T}'(1)\right]$$

$$Q(0) = 0$$

Eqn. (23) has the general solution

$$Q(r) = A\sinh[(1+i)\lambda r] + B\cosh[(1+i)\lambda r]\,.$$

Applying the boundary conditions gives

$$Q(r) = 3(\gamma-1)\left[\tilde{R} + \frac{i}{2\lambda^2}\tilde{T}'(1)\right]\frac{\sinh[(1+i)\lambda r]}{\sinh[(1+i)\lambda]}\,. \tag{24}$$

We now use $\Theta = Q/r$ and Eqn. (21) to write

$$\tilde{T} = 3(\gamma-1)\left[\tilde{R} + \frac{i}{2\lambda^2}\tilde{T}'(1)\right]\left(\frac{\sinh[(1+i)\lambda r]}{r\sinh[(1+i)\lambda]} - 1\right) \tag{25}$$

An expression for $\tilde{T}'(1)$ is found by taking the derivative of Eqn. (25) at r=1.

$$\tilde{T}'(1) = \frac{3(\gamma-1)\left[(1+i)\lambda\coth[(1+i)\lambda] - 1\right]}{1 - \frac{3(\gamma-1)i}{2\lambda^2}\left[(1+i)\lambda\coth[(1+i)\lambda] - 1\right]}\tilde{R}\,. \tag{26}$$

From Eqns. (18) and (26) we have an expression for the pressure inside the gas bubble at the wall.

$$i\Omega\tilde{P} = -i3\gamma\Omega\tilde{R} + \frac{3\gamma}{\mathrm{Pe}}\tilde{T}'(1)$$

$$\tilde{P} = -3\gamma\tilde{R} - i\frac{3\gamma}{2\lambda^2}\left[\frac{3(\gamma-1)\Psi(\lambda)}{1 - i\frac{3(\gamma-1)}{2\lambda^2}\Psi(\lambda)}\right]\tilde{R}\,. \tag{27}$$

Here

$$\Psi(\lambda) = (1+i)\lambda\coth[(1+i)\lambda] - 1\,.$$

In the limit of large Pe, $\lambda \to \infty$ and Ψ simplifies to

$$\Psi(\lambda) = (1+i)\lambda - 1\,.$$

Given the pressure inside the bubble as a function of frequency, it is possible to write down the response of an oscillating bubble from the Rayleigh-Plesset equation (1):

$$-\Omega^2 \tilde{R} + i2\zeta\Omega\tilde{R} + \frac{i\Omega \mathrm{Pe}}{3(\gamma-1)\Psi + i\Omega\mathrm{Pe}}\tilde{R} = \frac{1}{3\gamma} \tag{28}$$

This is the exact linear solution for an oscillating bubble. For the unforced case, the free response for a bubble has a frequency which is a root of:

$$-\Omega^2 + i2\zeta\Omega + \frac{i\Omega\mathrm{Pe}}{3(\gamma-1)\Psi + i\Omega\mathrm{Pe}} = 0 \tag{29}$$

The thermal effects for an oscillating bubble are contained in the last term in Eqn. (29), which comes from the pressure inside the bubble and can be rewritten more clearly as

$$\Re\left[\frac{i\Omega\mathrm{Pe}}{3(\gamma-1)\Psi + i\Omega\mathrm{Pe}}\right] - \Omega^2 + i\,2\left(\zeta + \frac{1}{2}\Im\left[\frac{i\Omega\mathrm{Pe}}{3(\gamma-1)\Psi + i\Omega\mathrm{Pe}}\right]\right)\Omega = 0$$

The real part of the pressure term provides the natural frequency of the bubble. This natural frequency includes a shift that takes into account thermal effects. One half the imaginary part of the pressure term is the contribution to damping by heat conduction. For comparison, the damping coefficient due to liquid viscosity, ζ, near resonance is 1.27×10^{-3} for a 100 micron bubble in water ($10°C$, 1 atm). The damping coefficient due to thermal effects is 42.7×10^{-3}.

The solution for Ω from Eqn. (29) can be expanded for both large and small Pe. For large Pe define a small parameter, ϵ, as

$$\epsilon = \frac{1}{\sqrt{\mathrm{Pe}}}$$

and take Ω to be

$$\Omega(\mathrm{Pe}\to\infty) = \Omega^{\infty} = \Omega_0 + \epsilon\,\Omega_1 + \epsilon^2\,\Omega_2 + \ldots \tag{30}$$

After taking the expansion described in Eqn. (30) and substituting it into Eqn. (29), we obtain the following set of algebraic equations:

$$\mathcal{O}(1): \qquad 1 - \Omega_0^2 = 0$$

$$\mathcal{O}(\epsilon): \qquad -2\,\Omega_0\,(\Omega_1 - i\,\bar{\zeta}) - \frac{3(\gamma-1)\,(1-i)}{\sqrt{2\Omega_0}} = 0$$

$$\mathcal{O}(\epsilon^2): \qquad -\Omega_1^2 - 2\,\Omega_0\,\Omega_2 + i\,2\,\Omega_1\,\bar{\zeta} - i\,\frac{3(3\gamma-2)\,(\gamma-1)}{\Omega_0} + \frac{3(\gamma-1)\,(1-i)\,\Omega_1}{(2\Omega_0)^{3/2}} = 0$$

Note that we scale ζ with the perturbation parameter, ϵ, so that $\zeta = \bar{\zeta}\,\epsilon$. Taking the leading order solution $\Omega_0 = 1$, the first two corrections of Ω^{∞} are

$$\Omega_1 = -\frac{3(\gamma-1)}{2\sqrt{2}} + i\left(\bar{\zeta} + \frac{3(\gamma-1)}{2\sqrt{2}}\right)$$

$$\Omega_2 = -\frac{1}{8}(i+1)\left(3(1+i)\,(3\gamma-1)\,(\gamma-1) - 3\sqrt{2}\,(\gamma-1)\bar{\zeta} + 2(1+i)\,\bar{\zeta}^2\right)$$

For small Pe the complete expansion of Ω is thus:

$$\Omega(\mathrm{Pe} \to 0) = \Omega^0 = \omega_0 + \epsilon\, \omega_1 + \epsilon^2\, \omega_2 + \dots \tag{31}$$

in which the small parameter, ϵ is now

$$\epsilon = \mathrm{Pe}$$

Eqn. (29) with Eqn. (31) yields the algebraic equations

$$\begin{aligned}
\mathcal{O}(1): &\qquad \frac{1}{\gamma} - \omega_0^2 = 0 \\
\mathcal{O}(\epsilon): &\qquad i\left(2\,\bar{\zeta} + \frac{\gamma - 1}{15\,\gamma^2}\right) - 2\,\omega_1 = 0 \\
\mathcal{O}(\epsilon^2): &\qquad \frac{(3\,\gamma + 7)\,(\gamma - 1)\,\omega_0^2}{1575\,\gamma^3} - 2\,\omega_0\,\omega_2 + i\,\frac{\omega_1\,(\gamma - 1)}{15\,\gamma^2} + i\,2\,\bar{\zeta}\,\omega_1 - \omega_1^2
\end{aligned}$$

Taking the leading order solution $\omega_0 = \frac{1}{\sqrt{\gamma}}$, the first two corrections to Ω^0 are

$$\begin{aligned}
\omega_1 &= i\left(\bar{\zeta} + \frac{\gamma - 1}{30\,\gamma^2}\right) \\
\omega_2 &= \frac{(\gamma - 1)\,(\gamma + 7)}{2520\,\gamma^{7/2}} - \frac{\bar{\zeta}\,(\gamma - 1)}{30\,\gamma^{3/2}} - \frac{\sqrt{\gamma}\,\bar{\zeta}^2}{2} = 0
\end{aligned}$$

Since we are interested in the damping due to thermal effects and not viscosity, we will now set $\bar{\zeta} = 0$. The first two corrections for Eqn. (29) are known in terms of Pe for both large and small Pe numbers.

$$\Omega^\infty \approx \Omega_0 + \Omega_1\,\frac{1}{\sqrt{\mathrm{Pe}}} + \Omega_2\,\frac{1}{\mathrm{Pe}} \tag{32}$$

$$\Omega^0 \approx \omega_0 + \omega_1\,\mathrm{Pe} + \omega_2\,\mathrm{Pe}^2 \tag{33}$$

These expansions can be combined in a two-point Padé approximation of the form

$$\Omega(\mathrm{Pe}) \approx \widehat{\Omega}(\mathrm{Pe}) = \frac{a_0 + a_1\,\mathrm{Pe}^{1/2} + a_2\,\mathrm{Pe}}{1 + b_1\,\mathrm{Pe}^{1/2} + b_2\,\mathrm{Pe}} \tag{34}$$

With the Padé expression above, only the first two terms from the expansion in Eqns. (32)-(33) can be matched. Expanding Eqn. (34) for both large and small Peclet numbers gives

For Pe $\to 0$

$$\widehat{\Omega}(\mathrm{Pe}) \approx a_0 + (a_1 - a_0\,b_1)\,\sqrt{\mathrm{Pe}} + (a_2 - a_1\,b_1 + a_0\,(b_1^2 - b_2))\,\mathrm{Pe}$$

For Pe $\to \infty$

$$\widehat{\Omega}(\mathrm{Pe}) \approx \frac{a_2}{b_2} + \left(-\frac{a_2\,b_1}{b_2^2} + \frac{a_1}{b_2}\right)\frac{1}{\sqrt{\mathrm{Pe}}}$$

The coefficients from the Padé expression can be found by matching them to the coefficients from the solution expansions (32,33).

$$
\begin{aligned}
a_0 &= \omega_0 \\
a_1 &= -\frac{\Omega_1\,\omega_1\,\omega_0}{(\Omega_0-\omega_0)^2} \\
a_2 &= \frac{\Omega_0\,\omega_1}{\Omega_0-\omega_0} \\
b_1 &= -\frac{\Omega_1\,\omega_1}{(\Omega_0-\omega_0)^2} \\
b_2 &= \frac{\Omega_1}{\Omega_0-\omega_0}
\end{aligned}
$$

With these coefficients, $\widehat{\Omega}$ can be explicitly written as

$$
\widehat{\Omega} = \frac{60\,\gamma\,(1-i)\,+\,\left(6\sqrt{2\,\gamma}\,+\,3\sqrt{2}\,(\gamma-1)\right)\sqrt{\mathrm{Pe}}\,+\,\left(2\,(\sqrt{\gamma}+1)\,(1+i)\right)\,\mathrm{Pe}}{60\,\gamma^{3/2}\,(1-i)\,+\,\left(6\sqrt{2}\,\gamma\,+\,3\sqrt{2\,\gamma}\,(\gamma-1)\right)\sqrt{\mathrm{Pe}}\,+\,\left(2\,(\sqrt{\gamma}+1)\,(1+i)\right)\,\mathrm{Pe}}
$$

The expression $\widehat{\Omega}$ is a function of the Peclet number. We note that $\widehat{\Omega}$ represents an approximation of the root of the harmonic oscillator Eqn. (29) (for $\zeta=0$). Thus (29) can be rewritten approximately as:

$$
\left(\Omega-\widehat{\Omega}\right)\left(\Omega+\widehat{\Omega}^*\right)=0 \tag{35}
$$

Here $\widehat{\Omega}^*$ is the complex conjugate of $\widehat{\Omega}$. We denote the damping due to thermal effects by ζ_{th}. From (35) we can write

$$
\zeta_{th}=\widehat{\Omega}-\widehat{\Omega}^* \,=\, 2\,\Im\{\widehat{\Omega}\}\,.
$$

Fig. 1 is a plot of the magnitude of the response of a bubble being forced periodically. The plot is for a Pe= 25, corresponding roughly to a bubble with a radius of 25 μm. Shown in the plot are four curves representing solutions obtained from the "exact" equation (28), from the Padé approximation, and from the adiabatic and the isothermal limits of Eqn. (28). For a Pe= 25 the plot shows that the adiabatic solution underdamps the response while the isothermal solution overdamps. Our Padé approximation provides more accurate results for the range of Pe numbers that we are interested in corresponding to bubble radii between 5–100 μm.

3 The Non-Equilibrium Equation of State (no relative motion)

We now consider a dilute mixture of identical spherical gas bubbles, having radii $R(\mathbf{x},t)$, which are suspended in a liquid medium of density ρ_ℓ and viscosity μ. When a separation of length scales

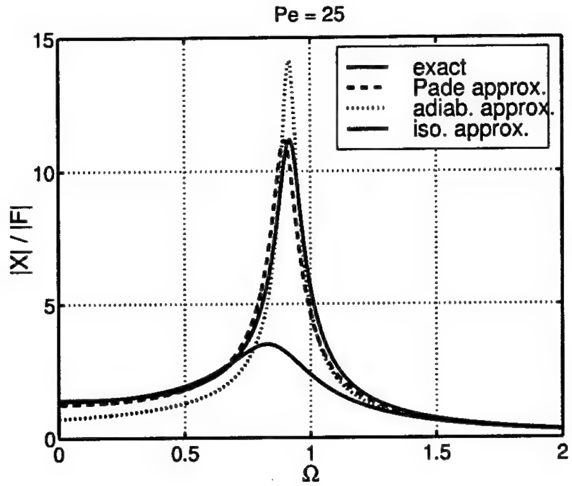

Figure 1:

holds such that the bubble radii are much smaller than the typical distance between bubbles, which is in turn smaller than the size of an averaging volume within which a large number of bubbles can be found, a non-equilibrium equation of state which relates pressure and density in the mixture can be obtained. This also requires the typical wavelength of sound waves in the mixture to be large compared to the size of the averaging volume, so that, to a good approximation, the acoustic pressure can be considered to be spatially uniform on the scale of the averaging volume. Under these approximations, the equation of motion for the radial oscillations of each of the identical noninteracting bubbles in the averaging volume is simply the Rayleigh-Plesset equation [32]

$$\rho_\ell[R\ddot{R} + \frac{3}{2}(\dot{R})^2] + 4\mu\frac{\dot{R}}{R} - P_o(\frac{R_o}{R})^{3\gamma} = -P(x,t)\,. \tag{36}$$

This is a nonlinear second order ODE for the bubble radius as a function of time. Here, each overdot represents a time-derivative, P_o and R_o are the equilibrium values of the pressure in the mixture (and in the bubbles) and the bubble radii, respectively, and $P(x,t)$ is the mean pressure in the mixture at the position of the averaging volume. Effects of surface tension have been neglected, while viscous damping of bubble pulsations has been taken into account. In (36), γ denotes the polytropic index of the gas inside the bubble; its value ranges from unity, for isothermal bubble oscillations, to the ratio of constant-pressure to constant-volume heat capacities of the gas for adiabatic oscillations. Generally in the Rayleigh-Plesset equation, the forcing pressure on the right-hand side is regarded as the pressure far from the individual bubble; due to the assumed separation of scales, this pressure is approximately the same as the mean pressure in the averaging volume surrounding the bubble.

The volume fraction ϕ of bubbles in the mixture can be related to their number density n (number of bubbles per unit volume) by

$$\phi = \frac{4}{3}\pi R^3 n\,. \tag{37}$$

In terms of ϕ the density of the bubbly mixture is given by

$$\rho = \rho_\ell(1-\phi) + \rho_g\phi \approx \rho_\ell(1-\phi)\,. \tag{38}$$

In the last approximation, we have neglected the contribution of the gas phase (density ρ_g) to the mass density of a dilute bubbly liquid. In the absence of bubble breakup and coalescence, the number of bubbles per unit mass of mixture remains constant with time, i.e.

$$\frac{n}{\rho} = \text{constant} = \frac{n_o}{\rho_o}\,. \tag{39}$$

The subscript 'o' refers to the equilibrium state. Eqs. (37) through (39) can now be combined to yield a one-to-one relationship between the mixture density ρ in the averaging volume and the radii of bubbles in that volume:

$$\frac{\rho}{\rho_\ell} = 1 - \phi_o(\frac{R}{R_o})^3\frac{\rho}{\rho_o}\,. \tag{40}$$

Upon solving Eq. (40) for R as a function of ρ and substituting the result into the Rayleigh-Plesset equation (36), one obtains a relationship of the general form

$$\begin{aligned} P \;=\; & P_o\left[\frac{\rho_\ell\,\phi_o\,\rho}{\rho_o(\rho_\ell-\rho)}\right]^\gamma + \frac{4\,\mu\,\rho_\ell}{3\,\rho(\rho_\ell-\rho)}\frac{D\rho}{Dt} + \\ & \frac{\rho_\ell\,R_o^2\,(\rho_o/\phi_o)^{2/3}}{3\,\rho^2\,(1/\rho-1/\rho_\ell)^{1/3}}\left[\frac{D^2\rho}{Dt^2} + \left(\frac{\rho_\ell}{6\,\rho(\rho_\ell-\rho)} - \frac{2}{\rho}\right)\left(\frac{D\rho}{Dt}\right)^2\right] \end{aligned} \tag{41}$$

Eq. (41) relates the mixture pressure P to the density ρ and its first two time-derivatives. Provided that translational motion of bubbles relative to the surrounding liquid is negligible, the time-derivatives in (41) are interpreted as material derivatives when the spatial variation of P and ρ on the macroscale is taken into account. A more in depth derivation of this fully nonlinear, non-equilibrium equation of state can be found in the paper by Nadim, Goldman and Barbone [33].

3.1 Traveling Wave Solution

In a continuum description of any fluid mixture in which direct viscous dissipation effects are negligible, the laws of conservation of mass and linear momentum in the continuum take the standard forms:

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x} \;=\; 0\,, \tag{42}$$

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}) + \frac{\partial p}{\partial x} \;=\; 0\,. \tag{43}$$

Here, ρ and p refer to the mean density and pressure in the mixture and u denotes the x-component of the translational velocity. Here, only one-dimensional motions along spatial coordinate x are considered and all three field variables are dependent upon (x,t) where t represents time.

One can seek a traveling wave solution of the above equations by defining the traveling-wave coordinate η as

$$\eta \equiv x - Ut\,. \tag{44}$$

U is the propagation speed of the waveform. A coordinate transformation from (x,t) to η reduces (42) and (43) to the coupled system of ODEs

$$(u-U)\,\rho' + \rho\,u' \;=\; 0\,, \tag{45}$$

$$\rho(u-U)\,u' + P' \;=\; 0\,. \tag{46}$$

A prime represents a total derivative with respect to η. Eqs. (45) and (46) can each be integrated once to yield

$$\rho\,(u-U) \;=\; C_1\,, \tag{47}$$

$$C_1\,u+P \;=\; C_2\,. \tag{48}$$

C_1 and C_2 are constants of integration.

We suppose that at large positive x, i.e., as η tends to infinity, the velocity u in the medium is zero and the pressure and density attain constant equilibrium values p_o and ρ_o, respectively. The integration constants C_1 and C_2 can be thus evaluated based upon conditions at infinity to be $C_1 = -\rho_o U$ and $C_2 = P_o$. When these values are substituted into (47) and (48) and the variable u is eliminated from the two equations, the result is a single nonlinear algebraic equation which relates the pressure and density profiles, $P(\eta)$ and $\rho(\eta)$:

$$P = P_o + \rho_o\,U^2\big(\frac{\rho-\rho_o}{\rho}\big)\,. \tag{49}$$

This relationship is exact for one-dimensional traveling waves in any continuum which is described by the standard continuity and Euler equations. For the bubbly liquids which are the subject of this study, one can find a further relationship, in the form of a non-equilibrium equation of state, between pressure and density (and its first two derivatives) in the mixture. The latter can be combined with the nonlinear algebraic equation (49) to reduce the problem to a single nonlinear ODE that is amenable to phase-plane analysis.

In the traveling-wave coordinate η, the equation of state (41) takes the form

$$\begin{aligned} P \;=\; & P_o\left[\frac{\rho_\ell\,\phi_o\,\rho}{\rho_o(\rho_\ell-\rho)}\right]^\gamma - \frac{4\,\mu\,\rho_\ell\,\rho_o\,U}{3\,\rho^2(\rho_\ell-\rho)}\,\rho' \\ & + \frac{\rho_\ell\,R_o^2\,(\rho_o/\phi_o)^{2/3}}{3\,\rho^2\,(1/\rho-1/\rho_\ell)^{1/3}}\left[\frac{\rho_o^2\,U^2}{\rho^2}\,\rho'' + \left(\frac{\rho_\ell}{6\,\rho(\rho_\ell-\rho)} - \frac{3}{\rho}\right)\frac{\rho_o^2\,U^2}{\rho^2}\,(\rho')^2\right] \end{aligned} \tag{50}$$

When this equation is combined with the nonlinear algebraic relation (49) resulting from the integration of continuity and Euler equations, a second-order nonlinear ODE is obtained for the density profile $\rho(\eta)$. The latter ODE is most simply given in its dimensionless form. For this purpose, we define dimensionless variables

$$\begin{aligned} \rho^* &\equiv \rho/\rho_o \\ P^* &\equiv P/P_o \\ \eta^* &\equiv \omega_o\eta/U \\ \rho_\ell^* &\equiv \rho_\ell/\rho_o = 1/(1-\phi_o) \end{aligned}$$

where quantities with superscript '$*$' are dimensionless. If this superscript is now dropped for notational simplicity, the resulting dimensionless nonlinear ODE for the density profile of the traveling wave is

$$1+\frac{\gamma}{\phi_o}\,M^2\left(\frac{\rho-1}{\rho}\right) = \left[\frac{\rho_\ell\,\phi_o\,\rho}{\rho_\ell-\rho}\right]^\gamma - \frac{\gamma\,\zeta\,\rho_\ell}{\rho^2(\rho_\ell-\rho)}\,\rho'$$

$$+\frac{\gamma\,\rho_\ell^{1/3}}{\phi_o^{2/3}\,\rho^{11/3}\,(\rho_\ell-\rho)^{1/3}}\left[\rho''+\left(\frac{\rho_\ell}{6\,\rho(\rho_\ell-\rho)}-\frac{3}{\rho}\right)(\rho')^2\right] \tag{51}$$

The terms that appear on the left-hand side of Eqn. (51) come from the integrated form of the Euler equations and the terms on the right-hand side are from the EOS. Two dimensionless parameters, ζ and M, fully characterize this system. These are defined by

$$\zeta \equiv \frac{4\,\mu}{\rho_\ell\,R_o^2\,\omega_o},$$

$$M \equiv U/c_o.$$

The parameter ζ represents a dimensionless damping constant while the Mach number M is the ratio of the speed of the traveling wave to the low-frequency sound speed in the bubbly liquid. In Eq. (51), a prime denotes a derivative with respect to the dimensionless traveling-wave coordinate η.

3.2 Phase-Plane Description of Traveling Waves

In order to explore the qualitative features of the solutions of Eq. (51), it is convenient to consider its phase-plane. The use of a phase-plane description in qualitatively describing traveling shock waves and the nature of the fixed points can also be found in Tan and Bankoff [7]; a similar phase-plane analysis of traveling waves in an Euler-Poisson model of a plasma is presented by Cordier et al. [34].

To examine the phase plane, the second-order nonlinear ODE is written as a system of two first-order equations by introducing the new variable g to denote ρ'. This results in the coupled system

$$\rho' = g, \tag{52}$$

$$\begin{aligned} g' &= A(\rho)\left[1+\frac{\gamma}{\phi_o}M^2\left(\frac{\rho-1}{\rho}\right)-\left(\frac{\rho_\ell\,\phi_o\,\rho}{\rho_\ell-\rho}\right)^{\gamma}+\frac{\zeta\,\gamma\,\rho_\ell}{\rho^2\,(\rho_\ell-\rho)}\,g\right] \\ &+ \left(\frac{3}{\rho}-\frac{\rho_\ell}{6\,\rho\,(\rho_\ell-\rho)}\right)g^2 \end{aligned} \tag{53}$$

Where,

$$A(\rho)=\frac{\phi_o^{2/3}\,\rho^{11/3}\,(\rho_\ell-\rho)^{1/3}}{\gamma\,\rho_\ell^{1/3}}$$

The phase-plane displays all possible solutions of this system of equations, with ρ on the horizontal axis and g (i.e. ρ') on the vertical axis. To find the fixed points of the system, we set both ρ' and g' to zero. The following equation describes the locations of the fixed points,

$$1+\frac{\gamma}{\phi_o}M^2\left(\frac{\rho-1}{\rho}\right)-\left[\frac{\rho_\ell\,\phi_o\,\rho}{\rho_\ell-\rho}\right]^{\gamma}=0. \tag{54}$$

Since $\rho_1 = 1$ satisfies Eqn. (54) it is a fixed point. Note that $\rho_\ell = 1/(1-\phi_o)$. The other fixed point can be found graphically by plotting Eqn. (54). Fig. (2) is a plot showing the two cases possible, $M > 1$ and $M < 1$. For $M > 1$ the solid curve crosses the ρ-axis at a value of ρ higher than unity. This is the location of the second fixed point. For $M < 1$ there is no second fixed point, indicated by the fact that the dashed curve never crosses the ρ-axis.

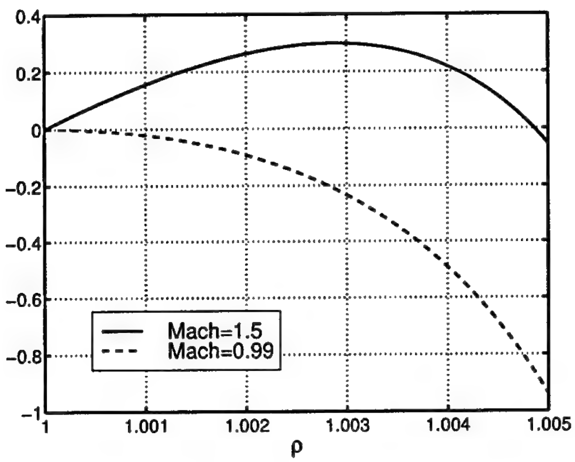

Figure 2:

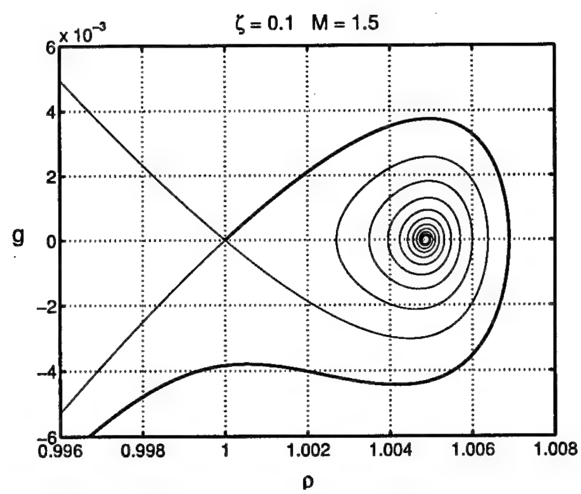

Figure 3:

Fig. 3 provides a typical phase portrait of the system for the case when Mach number exceeds unity — in this case $M = 1.5$. As seen in this figure, the fixed point at $\rho_1 = 1$ is a saddle point, with trajectories in the first and third quadrants moving away from this point and those in the second and fourth quadrants coming towards it, for increasing η. The other fixed point is a spiral node (when $\zeta = 0.1$). The location of the second fixed point, ρ_2, is the second root of Eqn. (54). There is a single trajectory which connects the two fixed points, identified in Fig. 3 as the shock solution.

This trajectory spirals away from the fixed point at ρ_2, and eventually tends toward the saddle point at $\rho = 1$, for increasing η. The solution profile $\rho(\eta)$ corresponding to this trajectory is displayed in Fig. 4.

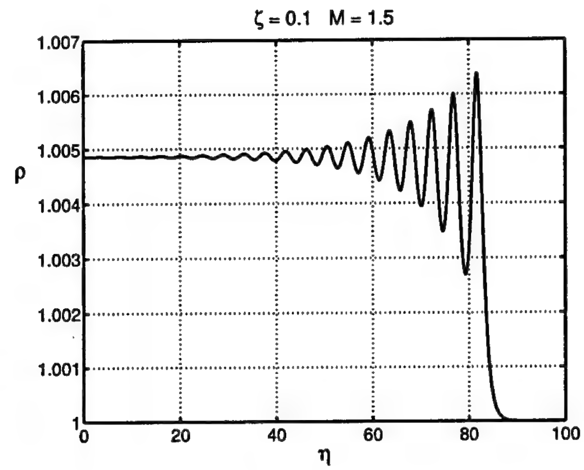

Figure 4:

This solution represents a density disturbance of steady shape that propagates to the right at dimensionless traveling-wave speed $M = U/c_o = 1.5$. The dimensionless density at large positive η approaches unity (i.e., the density approaches its equilibrium value ρ_o far ahead of the shock, as required by the original boundary conditions at large positive x). At large negative η, far behind the shock, the density settles down to the higher value ρ_2 after a series of oscillations which are associated with the spiral trajectory near the node in the phase plane. The fact that these oscillations exist at all is related to the presence of the second derivative term in ODE (51), which is in turn associated with the second derivative term in the non-equilibrium equation of state, (41), and the Rayleigh-Plesset equation (36). Thus, the oscillations observed behind shock waves in bubbly liquids are simply a result of the volume oscillations of gas bubbles in the medium when they are subjected to a somewhat sudden increase in pressure, as the pressure wave goes by.

It should be noted that the direction of the arrows in Fig. 3 corresponds to increasing η. The trajectories in the phase-plane can also be interpreted as functions of time but with their directions reversed. For instance, as a function of time, at a fixed spatial position in space initially far ahead of the shock, the density would start near its equilibrium value $\rho_1 = 1$ (the saddle fixed point) and, as time progresses, move along the shock trajectory (in the sense opposite to those indicated by the arrows) and end up at the higher density ρ_2 for large positive times.

4 Equation of State (with relative motion)

Relative motion has been shown in papers by Kameda and Matsumoto [10] and Ishii et al. [9] to be important in determining the exact waveform of shocks in bubbly liquids. It is possible to derive a new EOS that allows for relative motion between the gas phase (bubbles) and the liquid

phase. When there was no relative motion between the phases, it was possible to write the bubble radius, R, as a function only of the mixture density[33]. If we allow for relative motion, the exact functionality of R becomes more complicated. The number of bubbles that are in a given volume must simultaneously be tracked. This leads to a new relation for the bubble radius,

$$R(\rho, n) = \frac{\rho_\ell - \rho}{\frac{4\pi}{3}\rho_\ell n}. \tag{55}$$

Here n is called the number density and is defined as the number of bubbles per unit volume. Both ρ and n are field variables and depend on time. If Eqn. (55) is substituted into Eqn. (36), the resulting expression for the average pressure is

$$\begin{aligned} P(\mathbf{x}, t) \quad = \quad & \rho_\ell \left[\frac{n\ddot{\rho} + (\rho_\ell - \rho)\ddot{n}}{6(\frac{4\pi}{3}\rho_\ell)^{\frac{2}{3}} n^{\frac{5}{3}} (\rho_\ell - \rho)^{\frac{1}{3}}} - \frac{5\dot{n}\dot{\rho}}{9(\frac{4\pi}{3}\rho_\ell)^{\frac{2}{3}} n^{\frac{5}{3}} (\rho_\ell - \rho)^{\frac{1}{3}}} - \frac{11(\rho_\ell - \rho)^{\frac{2}{3}} \dot{n}^2}{18(\frac{4\pi}{3}\rho_\ell)^{\frac{2}{3}} n^{\frac{8}{3}}} \right. \\ & \left. + \frac{\dot{\rho}^2}{18(\frac{4\pi}{3}\rho_\ell)^{\frac{2}{3}} n^{\frac{2}{3}} (\rho_\ell - \rho)^{\frac{4}{3}}} \right] + 4\mu \frac{n\dot{\rho} + (\rho_\ell - \rho)\dot{n}}{3n(\rho_\ell - \rho)} + P_o \left(\frac{n(\rho_\ell - \rho_o)}{n_o(\rho_\ell - \rho)} \right)^\gamma , \end{aligned} \tag{56}$$

where the overdots represent the convective derivative with respect to the gas phase velocity. In Eqn. (56), the first term in the bracket is from inertial forces, the second term is due to viscous stresses on the bubble interface and the last term is from the ideal gas law for adiabatic bubbles. We now need equations describing how the number density, n, and the gas phase velocity, $\mathbf{v}$, change with space and time. For n we can use a continuity equation of number density as long as we assume that bubbles cannot be created or destroyed during our time of interest. For $\mathbf{v}$ we will use the gas phase momentum equation which reduces to a force balance on a bubble if the inertia of the gas is ignored. These equations are

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0. \tag{57}$$

$$\begin{aligned} 0 \quad = \quad & -\frac{3}{2}\rho_\ell \phi \frac{1}{R} \frac{D_g R}{Dt}(\mathbf{v} - \mathbf{u}) - \frac{1}{2}\rho_\ell \phi \left(\frac{D_g \mathbf{v}}{Dt} - \frac{D_\ell \mathbf{u}}{Dt} \right) \\ & -12\pi\mu_\ell R n(\mathbf{v} - \mathbf{u}) + \rho_\ell \phi \left(\frac{D_\ell \mathbf{u}}{Dt} - \mathbf{g} \right) \end{aligned} \tag{58}$$

The forces in Eqn. (58) were multiplied by n, to obtain the total force per unit volume. The first term in Eqn. (58) is the force resulting from changes in the bubble radius as it translates, and the last three terms in Eqn. (58) are due to the added mass, the drag and buoyancy, respectively. Note that in the force balance equation there exist two different material time derivatives, following either the bubble velocity, or the mixture velocity:

$$\begin{aligned} \frac{D_\ell}{Dt} &= \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \\ \frac{D_g}{Dt} &= \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \end{aligned}$$

If in Eqn. (58) buoyancy is neglected and Eqn. (55) is used to replace R, then we will have a equation describing the velocity of the gas phase in terms of the mixture density, mixture velocity and the number density:

$$\frac{D_g\mathbf{v}}{Dt} = 3\frac{D_\ell\mathbf{u}}{Dt} + (\mathbf{v}-\mathbf{u})\left[\frac{1}{\rho_\ell-\rho}\frac{D_g\rho}{Dt} + \frac{1}{n}\frac{D_gn}{Dt} - 18\nu_\ell\left(\frac{4\pi\rho_\ell n}{3(\rho_\ell-\rho)}\right)^{2/3}\right] \tag{59}$$

4.1 Traveling Wave Solution

In a manner similar to the case of no relative motion, we seek a traveling wave solution to the system of equations describing bubbly liquids with relative motion. As before, the governing equations are recast in terms of the traveling wave coordinate η, where $\eta = x - Ut$. The conservation equations are:

$$(u-U)\rho' + \rho u' = 0\,, \tag{60}$$

$$(v-U)n' + nv' = 0\,, \tag{61}$$

$$\rho(u-U)u' + P' = 0\,, \tag{62}$$

After integrating these equation we can assume for the boundary condition at $\eta \to \infty$ $(x \to \infty)$ that the velocities are zero and the pressure, mixture density and number density go to their equilibrium values of P_o, ρ_o and n_o, respectively. From these results we have equations relating the velocities and pressure to the density and number density:

$$u = U(\frac{\rho-\rho_o}{\rho}) \tag{63}$$

$$v = U(\frac{n-n_o}{n}) \tag{64}$$

$$P = P_o + \rho_o U^2(\frac{\rho-\rho_o}{\rho}) \tag{65}$$

Equations (63–65) can be combine with the traveling wave versions of Eqns. (59) and (56) to obtain two ODEs for $\rho(\eta)$ and $n(\eta)$. For brevity we only provide the system of ODEs in matrix form as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{\rho^3} - \frac{\rho-n}{n^2\rho(\rho_\ell-\rho)} & \frac{n-\rho}{n^3\rho} - \frac{1}{n^3} \\ 0 & 0 & n & \rho_\ell-\rho \end{bmatrix} \begin{Bmatrix} \rho' \\ n' \\ g' \\ q' \end{Bmatrix} = \begin{Bmatrix} g \\ q \\ f_1(\rho,n,g,q) \\ f_2(\rho,n,g,q) \end{Bmatrix} \tag{66}$$

All the variables in Eqn. (66) have been nondimensionalized using the same scales introduced in previous traveling wave analysis. In addition we have nondimensionalized velocities v and u with the constant traveling wave speed, U, and the number density, n, with $n_o = \phi_o/V_o$ where V_o is the initial volume of a bubble. As with the case of no relative motion, there are three main parameters controlling the qualitative structure of the shock wave. Fig. (5) shows the solutions to the ODEs

represented in Eqn. (66) for a viscous damping of $\zeta = 0.1$ and a range of values for the Mach number, M, and initial volume fraction of gas, ϕ_o.

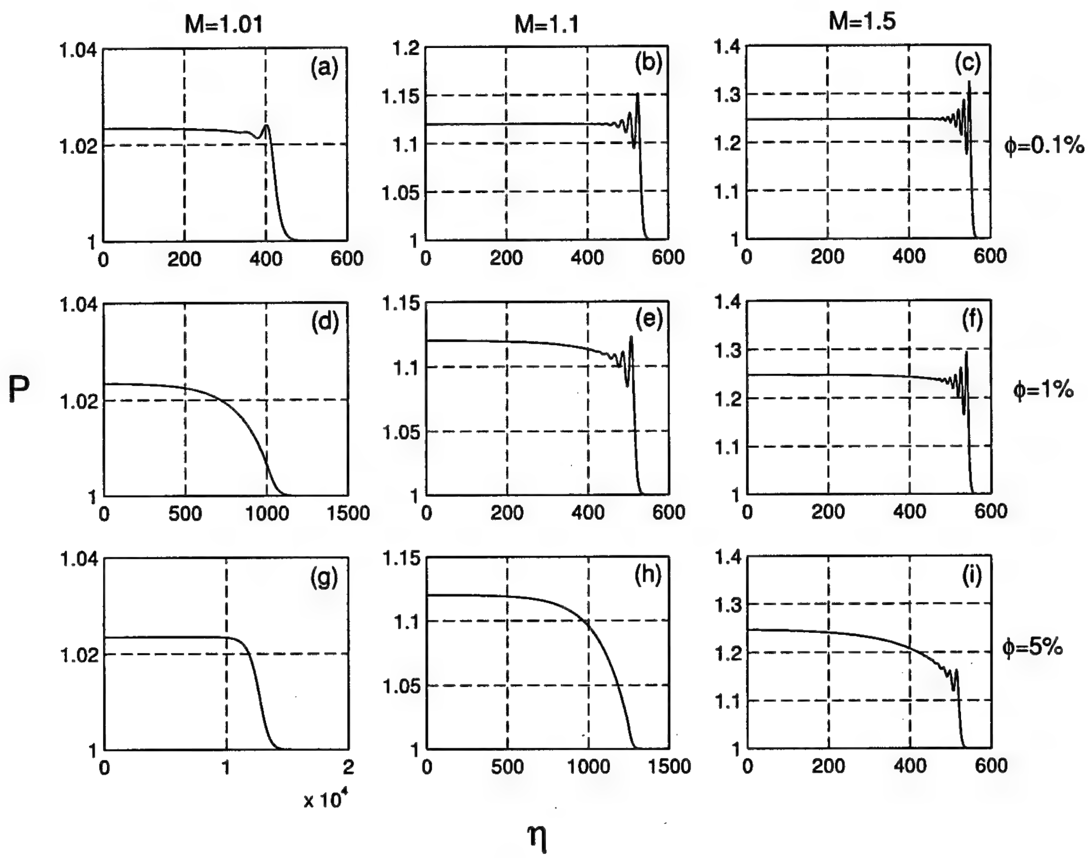

Figure 5:

The plots in Fig. (5) are arranged in a matrix form, with solutions of the same Mach number in columns and solutions with the same initial gas fraction (ahead of the shock) in rows. From the nine plots shown in Fig. (5) we can see three distinct shock waveforms. The shock in Fig. (5b) begins with a sharp rise in pressure to an over-peaked value followed by a oscillatory region. From Fig. (5f) we again see a sharp rise in pressure with an oscillatory region, but these oscillations are about a lower pressure value than the final equilibrium value. The oscillations are followed by a relaxation region which is due solely to the relative motion of the gas bubbles and not their oscillations (the bubbles have stopped oscillating by this point). The last qualitative waveform mentioned here is illustrated in Fig. (5g). This waveform results in a slower monotonic rise in pressure with no oscillatory behavior. The suppression of oscillations in the shock waveform was a direct result of allowing the bubbles to move relative to the liquid in the model. These results are consistent with the results of Ishii *et al* [9].

5 Conclusion

We have developed a non-equilibrium, nonlinear equation of state (EOS) that provides a dynamic relation between pressure in the mixture and the mixture density and the number density. The EOS contains the first two material time derivatives of both the mixture density and the number density, allowing for the possibility for solutions with oscillatory behavior. We have examined some of the possible traveling wave solutions obtained when the nonlinear EOS (with and without relative motion) is combined with the fully nonlinear equations of mass and momentum conservation. The structures of the shocks solutions were found to agree qualitatively with the waveforms observed in the experiments performed by Noordzij and van Wijngaarden.[2]

The attenuation of pressure waves are related to the damping mechanisms that exist for bubbles. The three most important damping mechanisms are associated with heat transfer between the gas bubble and the liquid, the drag due to relative motion of the bubble and viscous dissipation at the bubble interface. We have incorporated the latter two damping mechanism in our model. In addition, in our research we have shown how one can introduce an effective damping parameter that captures the thermal dissipation which occurs in a bubble oscillating periodically. We have shown that the effective damping parameter allows a more accurate description than that obtained by modeling the bubbles as either purely isothermal or adiabatic.

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13. ABSTRACT *(Maximum 200 words)*

Bubbly media play a significant role in underwater acoustics, medical ultrasound and in industrial systems where gas-liquid flows are present. The focus of our research has been to develop a continuum model for bubbly mixtures that can be used to model physical phenomena in these areas. The key to the continuum model is a nonlinear, non-equilibrium equation of state (EOS) that relates pressure to the mixture density and the number density (number of bubbles per unit volume) and their first two material time derivatives. The derivation of the EOS is presented here and a number of traveling wave solutions obtained using this nonlinear EOS are discussed. To develop an accurate model, two important damping mechanisms for the medium had to be incorporated: heat transfer and relative motion between the gas and liquid phases. To quantify the importance of heat transfer, an analysis of single-bubble radial oscillations was completed in this work, and a Padé approximation for the thermal damping was derived from the linearized gas dynamics equations. A second important damping mechanism arises from relative motion between the gas bubbles and the liquid. The quantitative effects of relative motion on the damping of waves in bubbly liquids has also been examined and is described here.

14. SUBJECT TERMS
Bubbly liquids, nonlinear waves, shocks

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